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PROOF OF SOME REMARKABLE RELATIONS IN THE METHOD OF LEAST SQUARES.

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I CONSIDER here the standard case in the method of least squares, treated for the case of equal weights in ANALYST, Vol. III, page 165 etc. Supposing now the observations of different weight, I will give a condensed view of all the formulæ required for a *complete* solution of this problem.

Let there be the linear obser ation equations:

$$a_1x+b_1y+\ldots.+l_1w-\mu_1=\Delta_1, \text{ weight } p_1, \qquad (1_1)$$
$$a_2x+b_2y+\ldots.+l_2w-\mu_2=\Delta_2, \quad " \quad p_2, \qquad (1_2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_mx+b_my+\ldots.+l_mw-\mu_m=\Delta_m, \quad " \quad p_m. \qquad (1_m)$$

From these, after applying the weights as stated, ANALYST, Vol III, p. 170, we obtain the normal equations:

$$[pa^2]x+[pab]y+\ldots.+[pal]w-[pa\mu]=[pa\Delta]=0, \qquad (2_x)$$
$$[pab]x+[pb^2]y+\ldots.+[pbl]w-[pb\mu]=[pb\Delta]=0, \qquad (2_y)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[pal]x+[pbl]y+\ldots.+[pl^2]w-[pl\mu]=[pl\Delta]=0. \qquad (2_w)$$

These equations give the unknown quantities $x, y, \ldots w$ in terms of $[pa\mu], [pb\mu], \ldots [pl\mu]$. To obtain the minimum sum of squares of the corrections we may use the formula:

$$[p\Delta^2]=-[p\mu\Delta]=-[pa\mu]x-[pb\mu]y-\ldots-[pl\mu]w+[p\mu^2], \qquad (3)$$

or since it is desirable to know the corrections $\Delta_1, \Delta_2, \ldots \Delta_m$, themselves, we may compute them by (1) and form $[p\Delta^2]$.

Since $x, y, \ldots w$ must be linear functions of $\mu_1, \mu_2, \ldots \mu_m$ we may assume

$$x=[\alpha\mu], \; y=[\beta\mu], \ldots w=[\lambda\mu]. \qquad (4)$$

These forms being introduced into the normal equations, these latter may be resolved into the following equations, since $\mu_1, \mu_2, \ldots \mu_m$ are independent,

$$[pa^2]\alpha_1+[pab]\beta_1+\ldots+[pal]\lambda_1-p_1a_1=0, \qquad (5_{x.1})$$
$$[pa^2]\alpha_2+[pab]\beta_2+\ldots+[pal]\lambda_2-p_2a_2=0, \qquad (5_{x.2})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[pa^2]\alpha_m+[pab]\beta_m+\ldots+[pal]\lambda_m-p_ma_m=0, \qquad (5_{x.m})$$
$$[pab]\alpha_1+[pb^2]\beta_1+\ldots+[pbl]\lambda_1-p_1b_1=0, \qquad (5_{y.1})$$
$$[pab]\alpha_2+[pb^2]\beta_2+\ldots+[pbl]\lambda_2-p_2b_2=0, \qquad (5_{y\,2})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[pab]\alpha_m+[pb^2]\beta_m+\ldots+[pbl]\lambda_m-p_mb_m=0, \qquad (5_{y.m})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[pal]a_1+[pbl]\beta_1+\ldots+[pl^2]\lambda_1-p_1l_1=0, \qquad (5_{w.1})$$
$$[pal]a_2+[pbl]\beta_2+\ldots+[pl^2]\lambda_2-p_2l_2=0, \qquad (5_{w.2})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[pal]a_m+[pbl]\beta_m+\ldots+[pl^2]\lambda_m-p_ml_m=0, \qquad (5_{w.m})$$

Summing products by $a_1, a_2, \ldots a_m$ then for each of the n sets we see that we must have

$$[aa]=1; \quad [a\beta]=0; \ldots [a\lambda]=0; \qquad (6_x)$$
$$[ba]=0; \quad [b\beta]=1; \ldots [b\lambda]=0; \qquad (6_y)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[la]=0; \quad [l\beta]=0; \ldots [l\lambda]=1. \qquad (6_w)$$

We have then, using equations (5) for forming these aggregates,

$$[aa]=[pa^2][a^2\div p]+[pab][a\beta\div p]+\ldots+[pal][a\lambda\div p]=1, \quad (7_{x.a})$$
$$[ba]=[pab][a^2\div p]+[pb^2][a\beta\div p]+\ldots+[pbl][a\lambda\div p]=0, \quad (7_{y.a})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[la]=[pal][a^2\div p]+[pbl][a\beta\div p]+\ldots+[pl^2][a\lambda\div p]=0, \quad (7_{w.a})$$
$$[a\beta]=[pa^2][a\beta\div p]+[pab][\beta^2\div p]+\ldots+[pal][\beta\lambda\div p]=0, \quad (7_{x.\beta})$$
$$[b\beta]=[pab][a\beta\div p]+[pb^2][\beta^2\div p]+\ldots+[pbl][\beta\lambda\div p]=1, \quad (7_{y.\beta})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[l\beta]=[pal][a\beta\div p]+[pbl][\beta^2\div p]+\ldots+[pl^2][\beta\lambda\div p]=0, \quad (7_{w.\beta})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[a\lambda]=[pa^2][a\lambda\div p]+[pab][\beta\lambda\div p]+\ldots+[pal][\lambda^2\div p]=0, \quad (7_{x.\lambda})$$
$$[b\lambda]=[pab][a\lambda\div p]+[pb^2][\beta\lambda\div p]+\ldots+[pbl][\lambda^2\div p]=0, \quad (7_{y.\lambda})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[l\lambda]=[pal][a\lambda\div p]+[pbl][\beta\lambda\div p]+\ldots+[pl^2][\lambda^2\div p]=1. \quad (7_{w.\lambda})$$

Multiplying the normal equations in their order by the quantities $[a^2\div p]$, $[a\beta\div p]$, $\ldots\ldots$ $[a\lambda\div p]$ which result from these equations and adding the products we obtain

$$x=[a\mu]=[a^2\div p][pa\mu]+[a\beta\div p][pb\mu]+\ldots+[a\lambda\div p][pl\mu], \quad (8_x)$$

Similarly we obtain

$$y=[\beta\mu]=[a\beta\div p][pa\mu]+[\beta^2\div p][pb\mu]+\ldots+[\beta\lambda\div p][pl\mu], \quad (8_y)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$w=[\lambda\mu]=[a\lambda\div p][pa\mu]+[\beta\lambda\div p][pb\mu]+\ldots+[\lambda^2\div p][pl\mu]. \quad (8_w)$$

These equations would have resulted as normal equations from the following equations as observation equations:

$$\frac{a_1}{p_1}\Big[pa\mu\Big]+\frac{\beta_1}{p_1}\Big[pb\mu\Big]+\ldots+\frac{\lambda_1}{p_1}\Big[pl\mu\Big]-\mu_1=\Delta_1, \text{ w't, } p_1, \quad (9_1)$$
$$\frac{a_2}{p_2}\Big[pa\mu\Big]+\frac{\beta_2}{p_2}\Big[pb\mu\Big]+\ldots+\frac{\lambda_2}{p_2}\Big[pl\mu\Big]-\mu_2=\Delta_2, \text{ “ } p_2, \quad (9_2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{a_m}{p_m}\Big[pa\mu\Big]+\frac{\beta_m}{p_m}\Big[pb\mu\Big]+\ldots+\frac{\lambda_m}{p_m}\Big[pl\,\mu\Big]-\mu_m=\Delta_m,\ \text{w't},\ p_m,\ (9_m)$$

by making $[p\Delta^2]$ = a minimum.

Resolving equations (8) we obtain the following formulæ analogous to (5);

$$a_1=[a^2\div p]\,p_1a_1+[a\beta\div p]\,p_1b_1+\ldots+[a\lambda\div p]\,p_1l_1,\quad(10_{x.1})$$

$$\beta_1=[a\beta\div p]\,p_1a_1+[\beta^2\div p]\,p_1b_1+\ldots+[\beta\lambda\div p]\,p_1l_1,\quad(10_{y.1})$$

$$\lambda_1=[a\lambda\div p]\,p_1a_1+[\beta\lambda\div p]\,p_1b_1+\ldots+[\lambda^2\div p]\,p_1l_1,\quad(10_{w.1})$$

Let r = probable error of an observation, weight 1, then $(r\div\sqrt{p})$ = the probable error of an observation, weight p, and applying the well known theorem [(43′), ANALYST, Vol. III, p. 140] we have the probable errors of the unknown quantities

$$r_x^2=\Big[\frac{a^2}{p}\Big]r^2;\ r_y^2=\Big[\frac{\beta^2}{p}\Big]r^2;\ldots r_w^2=\Big[\frac{\lambda^2}{p}\Big]r^2,\qquad(11)$$

and their reciprocal weights

$$\frac{1}{p_x}=\Big[\frac{a^2}{p}\Big];\quad\frac{1}{p_y}=\Big[\frac{\beta^2}{p}\Big];\ldots\frac{1}{p_w}=\Big[\frac{\lambda^2}{p}\Big].\qquad(12)$$

To obtain the probable error of $\mu_1+\Delta_1$, $\mu_2+\Delta_2$, . . . viz., the observed quantities, corrected, we can use either (1) or (9). By (1) we have, using (4),

$$\begin{aligned}\mu_1+\Delta_1&=a_1[a\mu]+b_1[\beta\mu]+\ldots+l_1[\lambda\mu]\\&=(a_1a_1+b_1\beta_1+\ldots+l_1\lambda_1)\mu_1\\&\quad+(a_1a_2+b_1\beta_2+\ldots+l_1\lambda_2)\mu_2\\&\quad+\ldots\\&\quad+(a_1a_m+b_1\beta_m+\ldots+l_1\lambda_m)\mu_m.\end{aligned}$$

Denoting the probable error of $\mu_1+\Delta_1$ by R_1 we have

$$\begin{aligned}R_1^2=\{\ &a_1^2[a^2\div p]+a_1b_1[a\beta\div p]+\ldots+a_1l_1[a\lambda\div p]\\&+a_1b_1[a\beta\div p]+b_1^2\ [\beta^2\div p]+\ldots+b_1l_1[\beta\lambda\div p]\\&+\ \ldots\\&+a_1\,l_1[a\lambda\div p]+b_1\,l_1[\beta\lambda\div p]+\ldots+l_1^2[\lambda^2\div p]\ \}r^2\\=(a_1a_1+b_1\beta_1+\ldots+l_1\lambda_1)&\,(r^2\div p_1),\ \text{by virtue of (10)},\end{aligned}\qquad(13)$$

and the reciprocal weight

$$\frac{1}{P_1}=\frac{1}{p_1}\Big(a_1a_1+b_1\beta_1+\ldots+l_1\lambda_1\Big).\qquad(14)$$

Likewise we have the expression for the correction Δ_1, by (1),

$$\begin{aligned}\Delta_1&=-\mu_1+a_1[a\mu]+b_1[\beta\mu]+\ldots+l_1[\lambda\mu]\\&=(-1+a_1a_1+b_1\beta_1+\ldots+l_1\lambda_1)\mu_1\\&\quad+(a_1a_2+b_1\beta_2+\ldots+l_1\lambda_2)\mu_2\\&\quad+\ldots\\&\quad+(a_1a_m+b_1\beta_m+\ldots+l_1\lambda_m)\mu_m.\end{aligned}$$

Denoting the probable error of Δ_1 by ρ_1 we have

$$\rho_1^2 = [1-2(a_1\alpha_1+b_1\beta_1+\ldots+l_1\lambda_1)](r^2\div p_1)+R_1^2$$
$$= (r^2\div p_1)-R_1^2, \text{ by (13), } = r_1^2-R_1^2, \qquad (15)$$

or $$r_1^2 = R_1^2+\rho_1^2, \qquad (16)$$

and $$1\div p_1 = 1\div P_1+1\div\pi_1, \text{ if } \pi_1 \text{ denotes the weight of } \Delta_1; \qquad (17)$$

hence the remarkable theorem:

The probable error of an observed quantity is the hypothenuse of a right angled triangle of which the probable error of the corrected quantity and that of the correction are the sides.

We have the weighted sum of equations like (13)

$$[pR^2] = ([a\alpha]+[b\beta]+\ldots+[l\lambda])r^2 = nr^2, \text{ by (6)}, \qquad (18)$$

or $$[p\div P] = n. \qquad (19)$$

Since $[p\div p] = m$ we have the theorem:

By the adjustment, the observed quantities are improved on the average in the ratio of n *to* m.

We have also $[p\rho^2] = (m-n)r^2$, (20); $[p\div\pi] = m-n$. (21)

We can place $[p\varepsilon^2] = [p\Delta^2]$ where $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_m$ are the mean errors of the corrections, but $[p\rho^2] = 2\rho^2[p\varepsilon^2]$ where $\rho = 0.47694$, the probable error for absolute precision, $= 1$. We have therefore from (20) the formula

$$r = \rho\sqrt{\left(\frac{2[p\Delta^2]}{m-n}\right)} = 0.6745\sqrt{\left(\frac{[p\Delta^2]}{m-n}\right)}; \qquad (22)$$

also the average probable error for an observed quantity

$$r_0 = r\sqrt{(m\div[p])}, \qquad (23)$$

and that of a corrected quantity $R_0 = r\sqrt{(n\div[p])}$. (24)

If it is required to find the probable error of

$$U = M+Ax+By+\ldots+Lw, \qquad (25)$$

we have, in terms of $\mu_1, \mu_2, \ldots \mu_m$,

$$U = M+(A\alpha_1+B\beta_1+\ldots+L\lambda_1)\mu_1$$
$$+(A\alpha_2+B\beta_2+\ldots+L\lambda_2)\mu_2$$
$$+ \quad . \quad . \quad . \quad . \quad .$$
$$+(A\alpha_m+B\beta_m+\ldots+L\lambda_m)\mu_m;$$

$$\therefore r_u^2 = \{A^2[\alpha^2\div p]+AB[\alpha\beta\div p]+\ldots+AL[\alpha\lambda\div p]$$
$$+AB[\alpha\beta\div p]+B^2[\beta^2\div p]+\ldots+BL[\beta\lambda\div p]$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+AL[\alpha\lambda\div p]+BL[\beta\lambda\div p]+\ldots+L^2[\lambda^2\div p]\}r^2. \qquad (26)$$

If U is not a linear function of $x, y, \ldots w$ we replace them by $x_0+\Delta x$, $y_0+\Delta y \ldots w_0+\Delta w$ and have, by Taylor's theorem,

$$U = U_0+\Delta U = f(x_0, y_0, \ldots w_0)+\frac{df}{dx}\Delta x+\frac{df}{dy}\Delta y+\ldots+\frac{df}{dw}\Delta w,$$
$$= f(x_0, y_0, \ldots w_0)-\frac{df}{dx}x_0-\frac{df}{dy}y_0-\ldots-\frac{df}{dw}w_0$$

$$+\frac{df}{dx}x+\frac{df}{dy}y+\ldots+\frac{df}{dw}w, \qquad (25')$$

which is of the form (25) and we have

$$r_u^2=\left\{\frac{df^2}{dx^2}\left[\frac{\alpha^2}{p}\right]+\frac{df}{dx}\cdot\frac{df}{dy}\left[\frac{\alpha\beta}{p}\right]+\ldots+\frac{df}{dx}\cdot\frac{df}{dw}\left[\frac{\alpha\lambda}{p}\right]\right.$$
$$+\frac{df}{dx}\cdot\frac{df}{dy}\left[\frac{\alpha\beta}{p}\right]+\frac{df^2}{dy^2}\left[\frac{\beta^2}{p}\right]+\ldots+\frac{df}{dy}\cdot\frac{df}{dw}\left[\frac{\beta\lambda}{p}\right]$$
$$+\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\left.+\frac{df}{dx}\cdot\frac{df}{dw}\left[\frac{\alpha\lambda}{p}\right]+\frac{df}{dy}\cdot\frac{df}{dw}\left[\frac{\beta\lambda}{p}\right]+\ldots+\frac{df^2}{dw^2}\left[\frac{\lambda^2}{p}\right]\right\}r^2. \qquad (26')$$

CONTINUATION OF THE NOTE AT P. 57 BY THE EDITOR.—Substituting for a its value $\frac{\omega^2 mR}{5F}+1$, also substituting for F its value, $m\frac{v^2}{D}$, from (3), and writing h for $\omega^2\div v^2$, we have

$$\theta=\frac{R}{r}\varphi-\frac{R}{r}\sqrt{\left(\frac{RDh}{5}\right)}\int\frac{d\varphi}{\sqrt{(\frac{1}{5}RDh+1-\cos\varphi)}};$$

or, substituting for $\cos\varphi$ its equivalent, $1-2\sin^2\frac{1}{2}\varphi$, dividing the quantity under the second radical by $\frac{1}{5}RDh$, placing this divisor in the denominator of the first radical and writing ψ for $\frac{1}{2}\varphi$ and c^2 for $10\div RDh$, we get

$$\theta=\frac{R}{r}\varphi-\frac{R}{r}\int\frac{2d\psi}{\sqrt{(1+c^2\sin^2\psi)}}.$$

From (3) we have

$$\theta=\frac{R}{r}\varphi-\frac{R}{r}\omega t,$$
$$\therefore\ t=\frac{2}{\omega}\int\frac{d\psi}{\sqrt{(1+c^2\sin^2\psi)}}. \qquad (8)$$

Substituting for $\sin^2\psi$ its equivalent $1-\cos^2\psi$, dividing the quantity under the radical by $1+c^2$, and writing e^2 for $c^2\div(1+c^2)$ we have

$$t=\frac{2}{\omega\sqrt{(1+c^2)}}\int\frac{d\psi}{\sqrt{(1-e^2\cos^2\psi)}}. \qquad (9)$$

The foregoing was sent to Mr. Charles H. Kummell, of the U. S. Lake Survey, with the request that he should calculate the numerical value of t; and, in response, Mr. Kummell has put the integral in the form

$$t=\int_{\frac{1}{2}\pi-\phi}^{\frac{\pi}{2}}\frac{d\Psi}{\sqrt{(a^2\cos^2\Psi+b^2\sin^2\Psi)}} \qquad (10)$$

which is more convenient for numerical calculation, and from which he finds

$$t_n=9^h05^m38^s.4.$$

Hence it appears that the ball will pass a point 180° from the initial point, $2^h54^m21^s.6$ befor the point A will pass the same point in space.